Variational Analysis of Interconnects & Power Grids based on Orthogonal Polynomial Expansions

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Why?

- Challenges
  - increasing Process Variations with Technology Scaling
  - significant uncertainty in circuit characteristics
- Consequences
  - impact on yield
  - circuit parameters are random variables (R, L, C)
  - circuit response is a Stochastic Process
  - every manufactured circuit - single manifestation of the process
- Obtain probability distributions of circuit delay, power

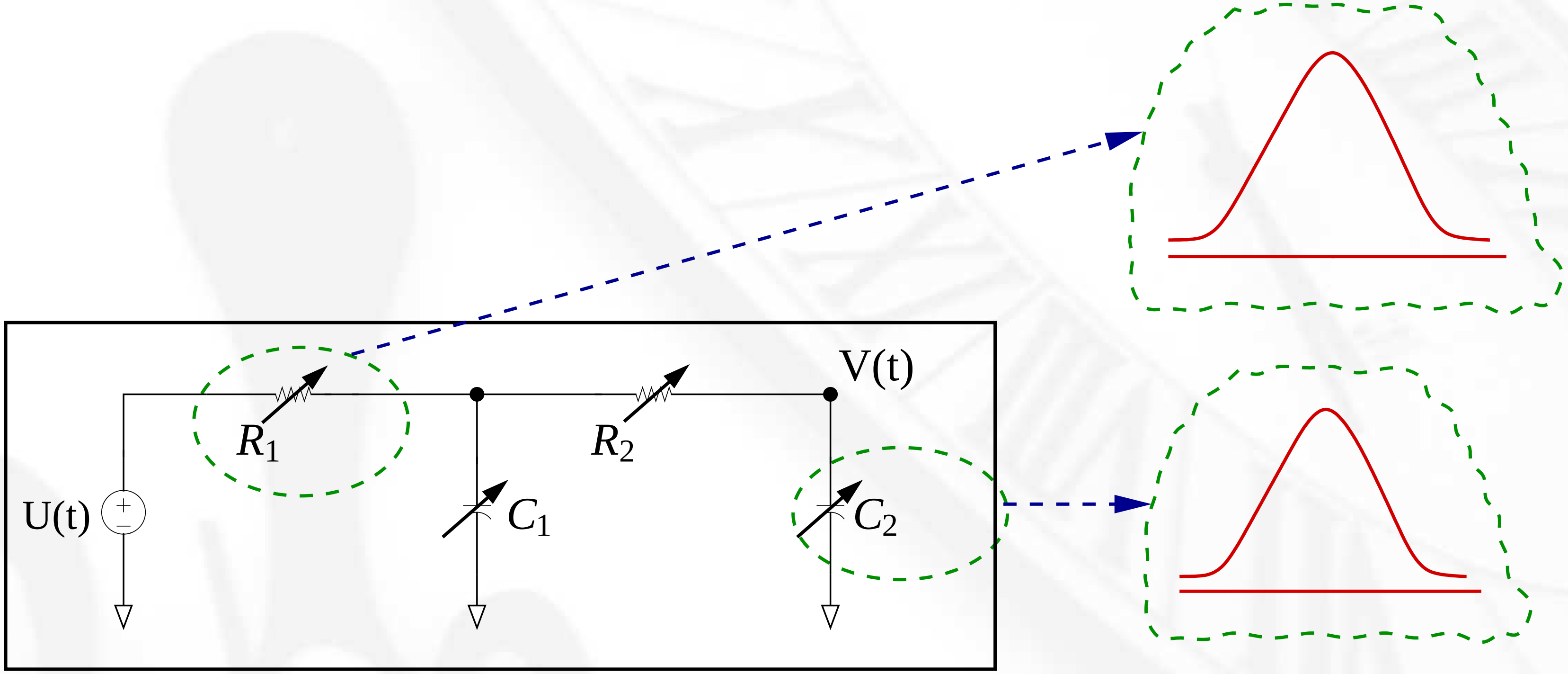
What do we need?

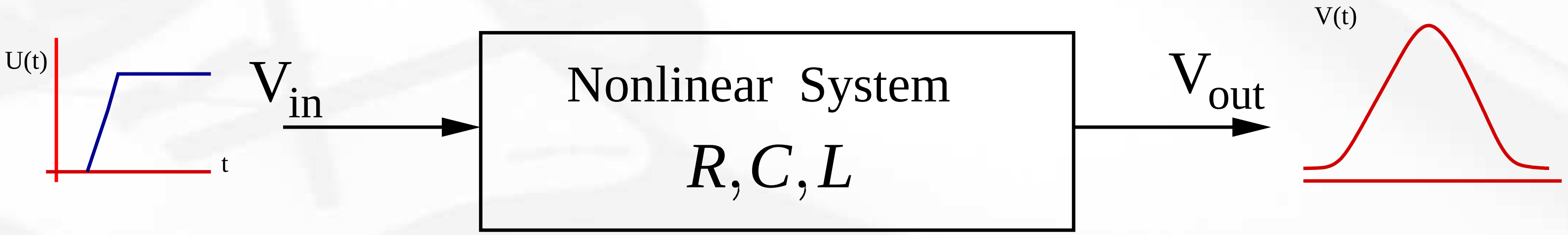
- Analytical models as functions of random variables for
  - interconnects
  - gates
  - power grids
- Handle various distributions of the random variables

Presentation Outline

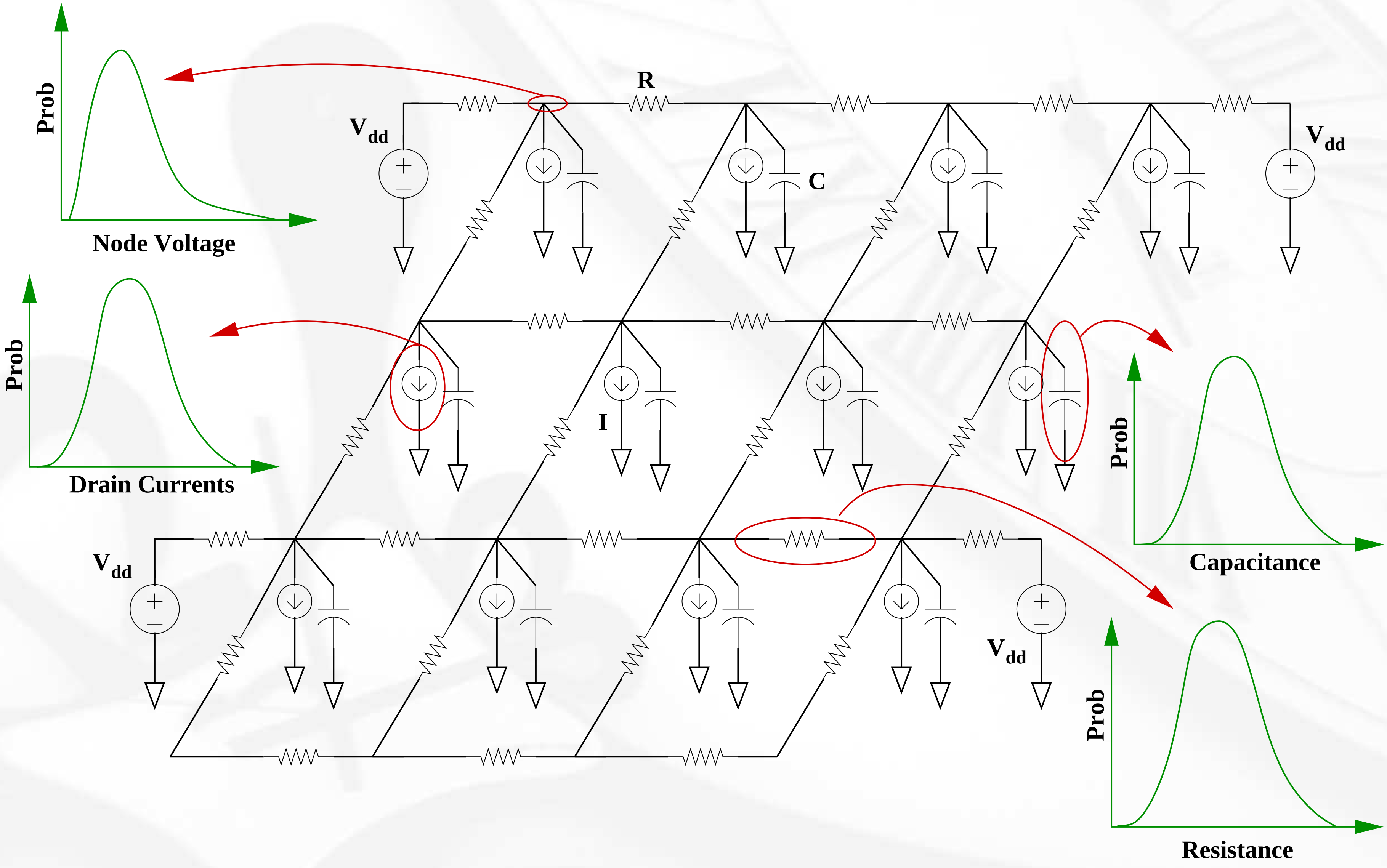
- Problem Formulation
- Summary of our Approach
- What's new?
- Details of our Approach
- Results
- Conclusions

Stochastic Interconnect Model

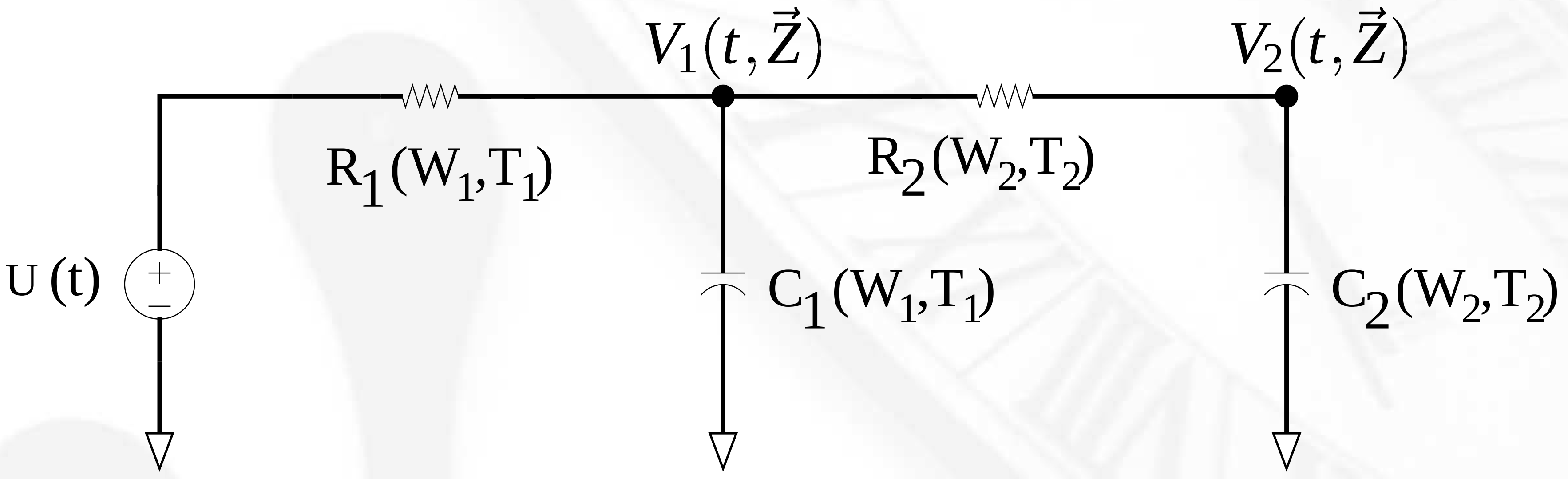




Stochastic Power Grid Model



Stochastic State-space Model



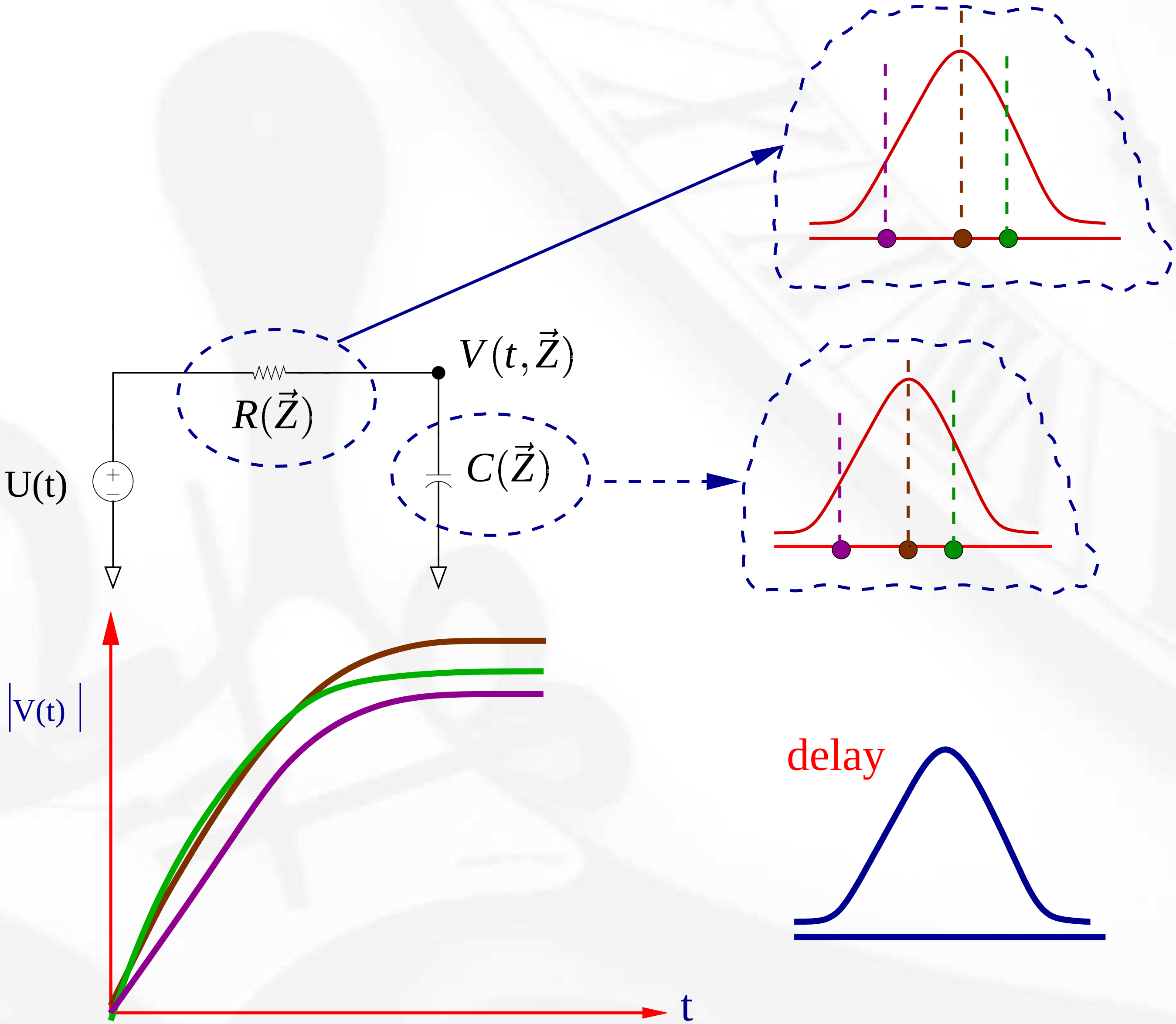
$$W_i = \overline{W_i} + \sigma(W_i)\, Z_{W_i}, \quad T_i = \overline{T_i} + \sigma(T_i)\, Z_{T_i}$$

$$\vec{Z} \;=\; (Z_{W_1}, Z_{T_1}, Z_{W_2}, Z_{T_2})$$

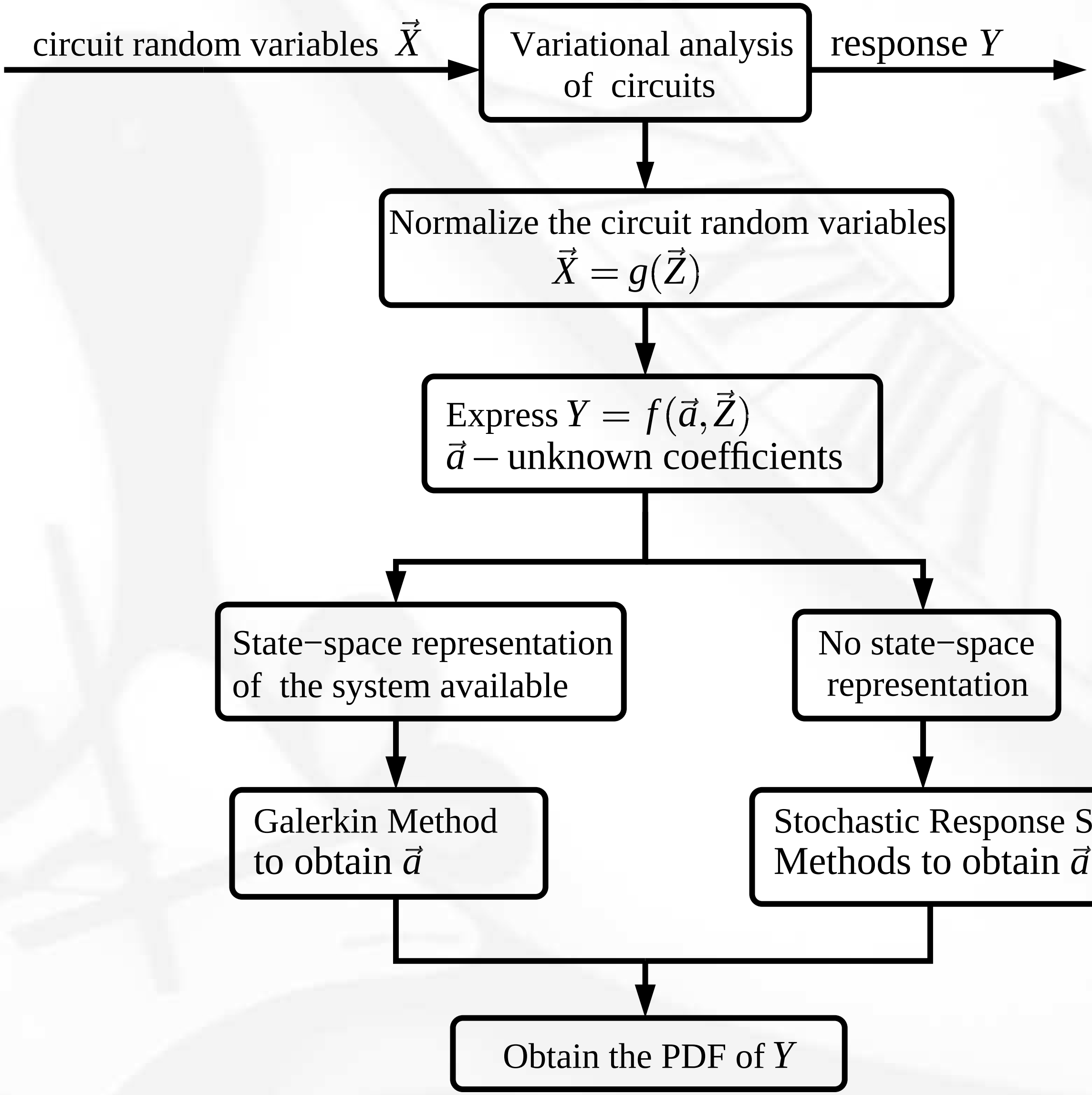
$$\vec{Z} = N(0,1), \quad \textbf{Mean}(\vec{Z}) \;=\; 0, \quad \textbf{Variance}(\vec{Z}) \;=\; 1$$

$$G(\vec{Z})\, V(t,\vec{Z}) + C(\vec{Z})\, \frac{dV(t,\vec{Z})}{dt} = G_1(\vec{Z})\, U(t)$$

Monte Carlo Simulations



Summary of Our Approach



What’s new?

- Explicit analytical expansion for stochastic response
- Accurate higher order expansions possible
- Optimal in the mean square sense
- Statistical interconnect analysis with process variations
  - perturbation based approaches (Liu et al., DAC’99, Heydari et al., ICCAD’01, Wang et al., DAC’04)
  - truncated balanced realizations based approaches (Philips, ICCAD’04)
  - interval based approaches (Ma et al., ICCAD’04)
- Statistical power grid analysis considering
  - input leakage current variations (Ferzli et al., ICCAD’03)
  - variations in the primary inputs (Pant et al., DAC’04)

Basis of Our Approach

- **Circuit response $V(t,\vec{Z})$ is 2^{nd} order stochastic process - finite variance, $\vec{Z} = N(0,1)$**
- **Consider a Hilbert space**

$$V(t,\vec{Z}) = \sum_{i=0}^{\infty} a_i(t)\, \Phi_i(\vec{Z})$$

- $\| V(t,\vec{Z}) - \sum_{i=0}^{\infty} a_i(s)\, \Phi_i(\vec{Z}) \| = 0$, $\| U, W \| = E(U\, W)$
- **$\{\Phi_i\}$ is complete set of orthonormal polynomials, called a basis**
- **Many alternatives for $\{\Phi_i\}$**

Hermite Polynomials

- **Hermite Polynomials - basis for Gaussian $\vec{Z}$**

$$H_n(\{i_1, i_2, \cdots, i_p\}) = (-1)^n e^{\frac{1}{2}\vec{Z}^t\vec{Z}} \frac{\partial^p}{\partial i_1 \partial i_2 \cdots \partial i_p} e^{-\frac{1}{2}\vec{Z}^t\vec{Z}}$$

- **where** $\vec{Z} = (Z_{i_1},\ Z_{i_2}, \ldots, \infty)$
- $\{i_1, i_2, \cdots, i_p\}$ **- any** p **variables from** $\vec{Z}$ **allowing repetitions**

order 0: $H_0(\{\}) = 1$,

order 1: $H_1(Z_1) = Z_1$, $H_1(Z_2) = Z_2$,

order 2: $H_2(Z_1, Z_1) = {Z_1}^2 - 1$, $H_2(Z_1, Z_2) = Z_1 Z_2$,

$H_2(Z_2, Z_2) = {Z_2}^2 - 1$

Hermite Polynomial Expansion

$$
\begin{aligned}
V(t,\vec{Z}) &= c_0(t)\,H_0 + \sum_{i_1=1}^{\infty} c_{i_1}(t)\,H_1(Z_{i_1}) \\
&+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} c_{i_1 i_2}(t)\,H_2(Z_{i_1},Z_{i_2}) \\
&+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3}(t)\,H_3(Z_{i_1},Z_{i_2},Z_{i_3}) \\
&+ \ldots\ldots\ldots
\end{aligned}
$$

Transformations to Gaussian

Distribution Type	Transformation
Uniform(a, b)	$a+(b-a)\left(\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{Z}{\sqrt{2}}\right)\right)$
Normal(μ, σ)	$\mu+\sigma\cdot Z$
Lognormal(μ, σ)	$\exp(\mu+\sigma\cdot Z)$
Gamma(α, β)	$\alpha\beta\left(Z\sqrt{\frac{1}{9a}}+1-\frac{1}{9a}\right)^3$
Exponential(λ)	$-\frac{1}{\lambda}\log\left(\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{Z}{\sqrt{2}}\right)\right)$

Best Orthonormal Bases

	Variable Distribution	Polynomials
Continuous	Gaussian Log-normal Gamma Beta Uniform	Hermite Hermite Laguerre Jacobi Legendre
Discrete	Poisson Binomial Negative Binomial Hypergeometric	Charlier Krawtchouk Meixner Hahn

State-space Available - Galerkin Method

$$V(t,\vec{Z}) = \sum_{i=0}^{\infty} a_i(t)\,\Phi_i(\vec{Z})$$

$$\tilde{V}_p(t,\vec{Z}) = \sum_{i=0}^{N} a_i(t)\,\Phi_i(\vec{Z})$$

$$\Delta_p(t,\vec{Z}) = G(\vec{Z})\,\tilde{V}_p(s,\vec{Z}) + C(\vec{Z})\,\dot{\tilde{V}}_p(s,\vec{Z}) - U(t,\vec{Z})$$

- $\Delta_p(t,\vec{Z})$ **minimized when** $\Delta_p(\vec{Z}) \perp \tilde{V}_p(t,\vec{Z})$
- $\left\langle \Delta_p(t,\vec{Z}), \Phi_j(\vec{Z}) \right\rangle = 0, \quad j = 0,1,\ldots,N$
- $\langle U, W \rangle = E(U\,W)$

Galerkin Method - Illustration

$$G(Z_W, Z_T)\, V(t, Z_W, Z_T) + C(Z_W, Z_T)\, \dot{V}(t, Z_W, Z_T) = U(t, Z_W, Z_T)$$

$$\begin{aligned}
G(Z_W, Z_T) &= G_a + G_b\, Z_W + G_c\, Z_T \\
C(Z_W, Z_T) &= C_a + C_b\, Z_W + C_c\, Z_T \\
U(t, Z_W, Z_T) &= U_a(t) + U_b(t)\, Z_W + U_c(t)\, Z_T
\end{aligned}$$

$$\begin{aligned}
V(t, Z_W, Z_T) &= a_0(t) + a_1(t)\, Z_W + a_2(t)\, Z_T + a_3(t)\, ({Z_W}^2 - 1) \\
&+ a_4(t)\, (Z_W\, Z_T) + a_5(t)\, ({Z_T}^2 - 1)
\end{aligned}$$

$$\begin{aligned}
\Delta_p(t, Z_W, Z_T) &= G(Z_W, Z_T)\, V(t, Z_W, Z_T) + C(Z_W, Z_T)\, \dot{V}(t, Z_W, Z_T) \\
&- U(t, Z_W, Z_T)
\end{aligned}$$

Galerkin Method - Illustration

- $\tilde{G}\, a(t) + \tilde{C}\, \dot{a(t)} \;=\; \tilde{b}$
- **Stochastic ($n \times n$) $\rightarrow$ ($6n \times 6n$) deterministic system**

$$
\begin{aligned}
V(t, Z_W, Z_T) \;&=\; a_0(t) + a_1(t)\, Z_W + a_2(t)\, Z_T + a_3(t)\, ({Z_W}^2 - 1) \\
&+\; a_4(t)\, (Z_W\, Z_T) + a_5(t)\, ({Z_T}^2 - 1)
\end{aligned}
$$

- $E[V(t, Z_W, Z_T)] = a_0(t)$
- $E[V^2(t, Z_W, Z_T)] = \sum_{i=0}^{5} {a_i}^2(t)$
- $E[V^n(t, Z_W, Z_T)] = E[V^{n-1}(t, Z_W, Z_T), V(t, Z_W, Z_T)]$
- **From moments, distribution can be estimated**

Variations in Inputs Only

$$G\,V(t,Z_W,Z_L)+C\,\dot{V}(t,Z_W,Z_L)=U(t,Z_W,Z_L)$$

$$\begin{aligned}U(t,Z_W,Z_L) &= U_0(t)+U_1(t)\,Z_W+U_2(t)\,Z_L+U_3(t)\,({Z_W}^2-1)\\ &+ U_4(t)\,Z_W\,Z_L+U_5(t)\,({Z_L}^2-1)\end{aligned}$$

$$\begin{aligned}V(t,Z_W,Z_L) &= V_0(t)+V_1(t)\,Z_W+V_2(t)\,Z_L+V_3(t)\,({Z_W}^2-1)\\ &+ V_4(t)\,(Z_W\,Z_L)+V_5(t)\,({Z_L}^2-1)\end{aligned}$$

$$G\,V_n(t,Z_W,Z_L)+C\,\dot{V}_n(t,Z_W,Z_L)=U_n(t,Z_W,Z_L) \quad \textbf{for } n=0\ldots5$$

- **Single LU Factorization, solve for different R.H.S**

State-space Unavailable - SRSM Method

- **Stochastic Response Surface Methods - Delay** $d(\vec{Z})$

$$d(\vec{Z}) = \sum_{i=0}^{\infty} a_i \Phi_i(\vec{Z})$$

$$\tilde{d}_p(\vec{Z}) = \sum_{i=0}^{N} a_i \Phi_i(\vec{Z})$$

$$\Delta_p(\vec{Z}) = d(\vec{Z}) - \tilde{d}_p(\vec{Z}) = \sum_{N+1}^{\infty} a_i \Phi_i(\vec{Z})$$

- $\Delta_p(\vec{Z})$ **minimized when** $\Phi_i(\vec{Z}) = 0$ **for** $i = (N+1)....\infty$
- **Force delay to satisfy circuit response from simulation at Zeros of higher order polynomials** $\Phi_i(\vec{Z})$

SRSM Method

- $\tilde{d}_p(\vec{Z}) = \sum_{i=0}^{N} a_i \Phi_i(\vec{Z})$
- **Zeros of (p+1) order polynomials $\Phi_i(\vec{Z})$ for p^{th} order Truncation**
- **Zeros called Collocation points, number required is N**
- **$> N$ Collocation points available; choose points in high probability regions**
- **SRSM is sensitive to the choice of N points**
- **Choose $M = 2N$ to $3N$ points, use a Least Mean Square fit**

Computational Cost of Galerkin

- Stochastic $(\mathbf{n} \times \mathbf{n}) \rightarrow (\mathbf{r}^{\mathbf{p}}\mathbf{n} \times \mathbf{r}^{\mathbf{p}}\mathbf{n})$ deterministic
- r - # of random variables, p - expansion order
- p = 2 or p = 3 generally sufficient
- Some efficient solution methods:
  - block LU factorization
  - iterative conjugate gradient based methods with pre-conditioners
  - reduced order modeling linearizes complexity
- Up to two orders (100x) of speedup over Monte Carlo

Computational Cost of SRSM

- Simulating circuit $2N$-$3N$ times is the dominant step
- $N = r^p$, r - random variables, p - expansion order
- p = 2 or p = 3 generally sufficient
- Up to two orders of speedup over Monte Carlo

General Methodology - Galerkin Method

- Model circuit using State-space equations
- Represent circuit response as an infinite series
- Truncate the infinite series and minimize the error norm
- Solve resulting linear system to obtain unknown coefficients
- Obtain the distribution

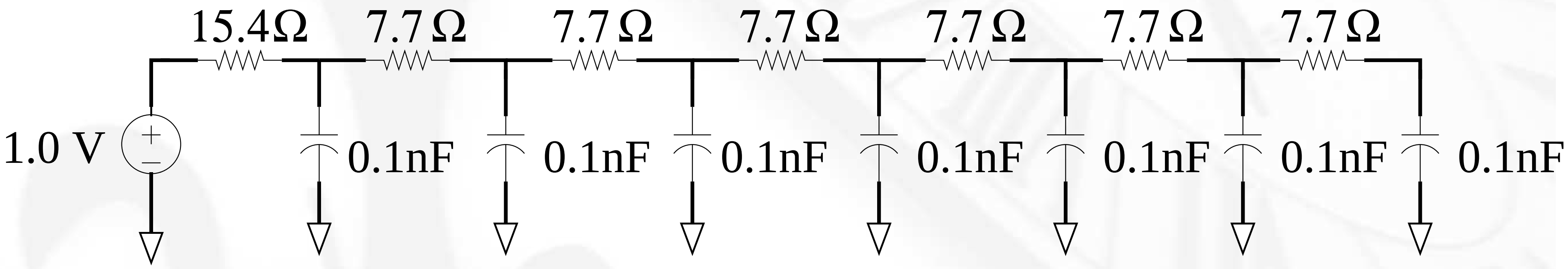
General Methodology - SRSM Method

- Approximate the circuit response by an order p expansion.
- Obtain the collocation points, zeros of (p+1) order polynomials
- Obtain the circuit response from simulation at collocation points
- Use regression to obtain a Least Mean Square fit for the response
- Obtain zeros of (p+2) order polynomials
- If error $> \varepsilon$, then repeat else stop

Experimental Results - Interconnects

- **OPERA - Orthogonal Polynomial Expansions for Response Analysis**
- **MC - Monte Carlo Simulations**
- **Gaussian Distribution**
- **Global Width (Z_W) and Thickness (Z_T) variations**
- **Linear models for G,C in Z_W, Z_T**
- **Order 2 OPERA-Galerkin and OPERA-SRSM**

Gaussian: RC tree 7 nodes



Gaussian: RC tree 7 nodes

- 50 % V_{DD} delay in ns. from OPERA, MC (1000 points)
- max. width variation 35 %, max. thickness variation 30 %

Node	MC μ	Galerkin μ	SRSM μ	MC 3σ	Galerkin 3σ	SRSM 3σ
2	3.93	3.92	3.92	0.567	0.573	0.544
3	6.79	6.78	6.78	0.968	0.979	0.955
4	9.31	9.30	9.30	1.320	1.337	1.286
5	11.26	11.25	11.25	1.592	1.613	1.590
6	12.61	12.62	12.62	1.782	1.805	1.726
7	13.40	13.40	13.40	1.890	1.917	1.884

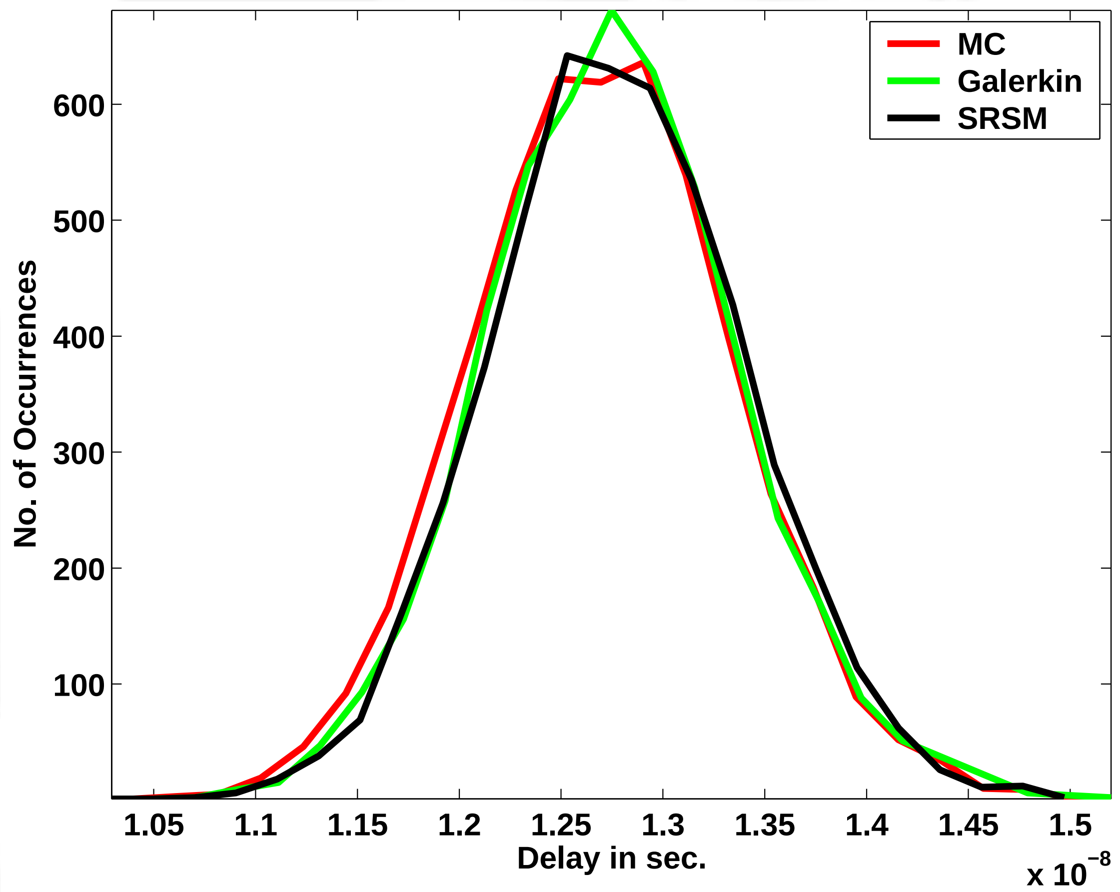
Gaussian: H-shaped clock tree (0.13 μ)

- 50 % V_{DD} terminal node delays for varying fanouts
- From OPERA and MC (1000 points)

# fanouts	MC μ	Galerkin μ	SRSM μ	MC 3σ	Galerkin 3σ	SRSM 3σ	Speed-up (X) Galerkin	Speed-up (X) SRSM
4	263.7	264.2	263.7	30.84	30.88	29.88	0.5	45
16	258.9	259.3	259.2	30.40	30.43	30.50	0.6	67
64	275.8	276.3	276.3	31.56	32.63	31.23	1.2	55
256	285.8	285.8	285.8	33.31	33.24	32.61	3.3	51
1024	296.5	296.2	296.2	34.68	34.88	35.98	10	52

RC Interconnect

- **Delay distribution from MC, OPERA**
- **max. width variation of 35% and max. thickness variation 30%**



Experimental Results - Power Grids

- **OPERA - Orthogonal Polynomial Expansions for Response Analysis**
- **MC - Monte Carlo Simulations**
- **Gaussian Distribution**
- **Global Width (Z_W) and Thickness (Z_T) variations of conductors**
- **Effective channel length variations ($Z_{L_{eff}}$) in the devices**
- 3σ **variations in** Z_W **= 20%,** Z_T **= 15%,** $Z_{L_{eff}}$ **= 20%**
- **Linear models for** G,C **and drain currents in** $Z_W, Z_T, Z_{L_{eff}}$

Power Grid Results

- 50 % V_{DD} delay in ns. from OPERA, MC (1000 points)
- Order 2 OPERA-Galerkin and OPERA-SRSM

Size	% Error in μ vs. MC ($\times 10^{-2}$)		% Error in σ vs. MC		Speed-up (X)	
(#nodes)	Galerkin	SRSM	Galerkin	SRSM	Galerkin	SRSM
19181	1.550	1.743	2.530	3.167	101	38
25813	4.222	0.250	3.410	5.325	20	28
34938	2.040	0.500	1.530	0.967	65	38
49262	1.992	1.167	6.730	2.961	27	32
62812	6.800	0.900	3.820	2.130	85	37
91729	1.370	0.300	3.280	4.277	124	38
351838	9.260	1.279	5.270	4.853	104	38

Conclusions

- Novel scheme for variational analysis
- Explicit functional representation of stochastic circuit response
- Optimal in the mean square sense
- Extensively verified for several test cases
- Excellent match with MC simulations
- Speed-ups of up to two orders of magnitude over MC